4491. Proposed by Lorian Saceanu.

Let a, b, c be the side lengths of acute-angled triangle ABC lying opposite of angles $\angle A, \angle B, \angle C$, respectively. Let r be the inradius of ABC and let R be its circumradius. Prove that

$$\frac{a\angle A + b\angle B + c\angle C}{a+b+c} \le \arccos\frac{r}{R}.$$

Solution by Arkady Alt, San Jose, California, USA.

Since function $\arccos x$ is concave down on $[0,1]$ ($(\arccos x)'' = -\frac{x}{\sqrt{(1-x^2)^3}}$)

and $\angle A = \arccos\frac{b^2+c^2-a^2}{2bc}, \angle B = \arccos\frac{c^2+a^2-b^2}{2ca}, \angle C = \arccos\frac{a^2+b^2-c^2}{2ab}$

then, applying Jensen's Inequality with weights (a, b, c), we obtain $\frac{a\angle A + b\angle B + c\angle C}{a+b+c} =$

$$\frac{\sum_{cyc} a\arccos\frac{b^2+c^2-a^2}{2bc}}{a+b+c} \le \arccos\left(\frac{\sum_{cyc} a\cdot\frac{b^2+c^2-a^2}{2bc}}{a+b+c}\right) = \arccos\left(\frac{\sum_{cyc} a^2(b^2+c^2-a^2)}{2abc(a+b+c)}\right)$$

$$\arccos\left(\frac{2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4}{2abc(a+b+c)}\right) = \arccos\left(\frac{16[ABC]^2}{2abc(a+b+c)}\right) =$$

$$\arccos\left(\frac{16r^2s^2}{2\cdot 4Rrs\cdot 2s}\right) = \arccos\left(\frac{r}{R}\right).$$